Math 254-2 Exam 10 Solutions

1. Carefully define the term "dependent". Give two examples in $\mathbb{R}^2$.

   A set of vectors is dependent if there is a nondegenerate linear combination yielding the zero vector. Examples in $\mathbb{R}^2$ include $\{(0,0)\}$, $\{(1,1),(2,2)\}$, $\{(1,0),(0,1),(2,3)\}$.

For the next two problems, consider the matrix $A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{pmatrix}$.

2. Calculate $|A|$ by using the formula for 3×3 determinants.

   We have $|A| = (2)(0)(-3)+(2)(2)(0)+(-1)(1)(1)-(0)(0)(-1)-(1)(2)(2)-(-3)(1)(2) = -1-4+6 = 1$

3. Calculate $|A|$ by expanding on the second column.

   We have $|A| = 2(-1)^{1+2}\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} + 0(-1)^{2+2}\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} + 1(-1)^{3+2}\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = -2(-3-0)-(4-(-1)) = -2(-3)-(5) = 1$.

4. Solve the linear system $\left\{ \begin{matrix} 2x+y & = 5 \\ -2x+y & = 1 \end{matrix} \right\}$ using Cramer's rule.

   Cramer's rule gives $x = \frac{\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{4}{4} = 1, y = \frac{\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{12}{4} = 3$.

5. Find $|B|$, for $B = \begin{pmatrix} 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 \\ 0 & 2 & 0 & -2 & 0 \\ 4 & 0 & 1 & 0 & 1 \\ 1 & 1 & -2 & 0 & 3 \end{pmatrix}$.

   Many approaches are possible. To do this efficiently requires a combination of elementary row/column operations and Laplace expansions. Adding the second column to the fourth gives the matrix $C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & -1 \\ 0 & 2 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 1 \\ 1 & 1 & -2 & 1 & 3 \end{pmatrix}$. $|B| = |C|$, and we find $|C|$ by expanding on the third row: $|C| = 2C_{32}$, where $C_{32} = (-1)^{3+2}|D|$, for $D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & -1 \\ 4 & 1 & 0 & 1 \\ 1 & -2 & 1 & 3 \end{pmatrix}$. We find $|D|$ by expanding on the first row: $|D| = (2)D_{11}$, where $D_{11} = (-1)^{1+1}|E|$, for $E = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 1 & 3 \end{pmatrix}$. We now calculate $|E| = 0+(-2)+(-1)-0-2-3 = -8$. Hence $|D| = -16, |C| = |B| = 32$.